

PDF's Of The Burgers Equation On The Semiline With Fluctuating Flux At The Origin

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Abstract

We derive the asymptotic behaviour of the one point probability density for the inhomogeneous shock slopes in the turbulent regime, when a Gaussian fluctuating flux at origin drives the system. We also calculate the time dependence of the x_f beyond which there won't exist any velocity shocks as $x_f \cong t^{3/4}(\log t)^{1/4}$. We argue that the stationary state of the problem would be equivalent with the long time limit of the diffusion equation with a random source at origin.

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1 Introduction

Burgers equation [1] initially was suggested as a model for describing the propagation of nonlinear waves in dissipative media [2]. Burgers equation with a stochastic source has been suggested as a phenomenological equation covering the behaviour of a large universality class of nonequilibrium growth models [3]. Under a nonlinear transformation it can also be transformed into a model for diffusion in Random landscapes which is itself related to a rich class of phenomena like, Directed Polymers in Random media [4], wetting [5], Random Magnets [6] and Pinning of vortex lines in superconductors [7]. This equation is thoroughly studied as an approximate model describing the formation of large scale structures in Universe [8]. Burgers Equation with a noise term has also been studied as a toy model for fully-developed turbulence with using a variety of methods [9]. The full description of this equation is still far from clear but the introduction of new methods has given a better understanding of the relationship between different realms of physics. The large time solutions of deterministic Burgers equation with random initial conditions has been investigated by the turbulence community for mimicking the behaviour of decaying turbulence [10]. An alternative way of introducing fluctuations is to consider random initial conditions. The physical relevance for instance comes from considering a variable injection of fluid into a semi-infinite slab of soil.

Semiline solutions of Burgers equation with a given deterministic flux at origin has been derived [11], which consistently reproduces the inhomogeneous shocks of Burgers equation at large spatial scales. By allowing random flux, we observe that shocks are present at large times but with random values and the velocities become ordered.

2 The Semiline Solution

Consider the Burgers equation on the semiline, with the initial and boundary condition:

$$u_t + uu_x = \nu u_{xx} \quad (1)$$

$$u(x, 0) = 0 \quad 0 \leq x \leq \infty \quad (2)$$

$$\frac{u^2(0, t)}{2} + \nu u_x(0, t) = g(t) \quad (3)$$

The above initial value problem is solved through a generalised Hopf-Cole transformation [11].

$$u(x, t) = \frac{2}{\sqrt{\pi}} \frac{\int_0^t dt' \frac{g(t')c(0)}{\sqrt{4\nu(t-t')}} \exp(-\frac{1}{\nu}(\frac{x^2}{4(t-t')} - \int_0^{t'} g(t'')dt''))}{c(t) - \frac{1}{\nu} \int_0^t dt' g(t') \exp(\frac{1}{\nu} \int_0^{t'} g(t'')dt'') \operatorname{erf}(\frac{x}{\sqrt{4\nu(t-t')}})}, \quad (4)$$

where:

$$c(t) = c(0)e^{\int_0^t g(t')dt'} \quad (5)$$

In the limit where ν tends to zero we can expand equation(4) for regions where $\frac{x}{\sqrt{4\nu t}} \gg 1$:

$$u(x, t) \cong \frac{2}{\sqrt{\pi}} \frac{\int_0^t dt' \frac{g(t')c(0)}{\sqrt{4\nu(t-t')}} \exp(-\frac{1}{\nu}(\frac{x^2}{4(t-t')} - \int_0^{t'} g(t'')dt''))}{1 + \frac{1}{\nu} \int_0^t dt' g(t') (\frac{\sqrt{4\nu(t-t')}}{\sqrt{\pi x}}) \exp(-\frac{1}{\nu}(\frac{x^2}{4(t-t')} - \int_0^{t'} g(t'')dt''))}, \quad (6)$$

Following Burgers[2], we note that the main contribution to the integral in (6) come from points which maximize the exponent:

$$h(x, t, t') = -\frac{x^2}{4(t-t')} + \int_0^{t'} g(t'') dt'' \quad (7)$$

After some manipulations, equation (6) reduces to

$$u(x, t) \cong \frac{x}{t-t_i} \quad (8)$$

where $t_i(x, t)$ are the absolute maxima of $h(x, t, t')$. These are indeed the shock waves discovered by Burgers through an elegant geometric construction.

Burgers geometric construction finds the maxima of $h(x, t, t')$ by defining two functions:

$$S(t') = \frac{x^2}{4(t-t')} + H, \quad (9)$$

$$G(t') = \int_0^{t'} g(t) dt, \quad (10)$$

where H is a constant. For a given t , one looks for the maximum value of H , called $H^*(t_i)$, where the hyperbola $S(t')$ just touches the curve $G(t')$. For large values of x the hyperbola $S(t')$ flattens, and it may touch $G(t)$ at two points. When this happens we have a shock wave, thus the solutions are comprised of linear parts, $\frac{x}{t-t'}$. In small regions of space, intervening the linear parts, the velocity profile is more complicated. The long time velocity profile is a series of shocks with reducing slopes. This can be understood as response of the system to impulses imported to it at origin, at consecutive times.

3 One Point Probability Density

Let us now analyse the same problem for the case of the random impulse at origin and assume that $g(t)$ is a random variable with Gaussian distribution:

$$\langle g(t) \rangle = 0 \quad (11)$$

$$\langle g(t)g(t') \rangle = 2\sigma_g^2 \Delta(t-t') \quad (12)$$

Where Δ is any reasonable correlation which decays with growing argument. This means that $G(t)$, as defined in equation(10) is a stochastic process and we assume that the variance of this process depends linearly on time so that it a nonhomogeneous process with some correlation time τ overwhich its values are correlated and beyond that uncorrelated. Giving address to the geometrical construcion discussed in the previous section the problem reduces to finding the maxima of a nonhomogeneous process $G(t)$. For this purpose we can adopt the methods previously used for the random initial conditions[10]. Consider the commulant probability of the event that the absolute maximum of $h(x, t, t')$ occurs in the interval $[t'_1, t'_2]$, that is,

$$F(H, [t'_1, t'_2]) = Prob\left(\frac{x^2}{4(t-t'_i)} + G(t'_i) < H; t'_i \in [t'_1, t'_2]\right) \quad (13)$$

If the correlation time of $G(t)$ is smaller than the mean difference of two adjacent t'_i 's:

$$\overline{(t'_{i+1} - t'_i)} > \frac{1}{\sigma_g^2} \quad (14)$$

then one can assume independence for regions inside the interval $[t_1, t_2]$ and outside it. Thus the probability of finding $t_i \in [t'_1, t'_2]$ is :

$$Prob(t' \in [t'_1, t'_2]) = \int_{-\infty}^{\infty} F(H, [0, t] - [t'_1, t'_2]) P(H, [0, t] - [t'_1, t'_2]) dH \quad (15)$$

where $P = \frac{dF}{dH}$.

Indeed in the Poissonian approximation we have :

$$F(H, [t_1, t_2]) \cong e^{-N(H, [t_1, t_2])} \quad (16)$$

where N is the expected number of crossings between $G(t')$ and $H + \frac{x^2}{4(t-t')}$ in the interval $[t_1, t_2]$ and is given

$$N(H, [t_1, t_2]) = \langle \int \delta(\Phi) \left\{ \frac{\partial \Phi}{\partial t} \right\}^+ dt' \rangle \quad (17)$$

where $\Phi = \frac{x^2}{4(t-t')} - G(t')$ and $\{x\}^+ = x\theta(x)$. The average is taken over the joint distribution of $g(t)$ and $G(t)$, and there is the constraint that at the point of intersection $G(t)$ has the positive slope. After some manipulations and in the limit when $\frac{x}{\sqrt{\sigma_g}} \ll t$ and $H \gg \sigma_g t$ we get:

$$N(H, [t_1, t_2]) \cong \int_{t_1}^{t_2} dt' \frac{\sqrt{\sigma_g^2 t'}}{H \sqrt{2\pi}} e^{-\frac{(H^2 + \frac{x^2}{4(t-t')})^2}{2t' \sigma_g^2}} \quad (18)$$

and consequently $F(H, [0, t])$ will become:

$$F(H, [0, t]) \cong e^{-\int_0^t dt' \frac{\sqrt{\sigma_g^2 t'}}{S(t') \sqrt{2\pi}} e^{-\frac{S(t')^2}{2t' \sigma_g^2}}} \quad (19)$$

where $S(t') = H + \frac{x^2}{4(t-t')}$. For calculating the exponent we use from the saddle point approximation in the integral of the exponent. So in the large time limit we simply have the following saddle point solution for t^* :

$$t - t^* \cong \frac{x^2}{4H} + \sqrt{\frac{tx^2}{4H}} \quad (20)$$

Before proceeding further we wish to find an estimate for x_f where the shock would disappear. Taking a closer look at the t^* we find that when $N(H, [0, t])$ is maximum around t^* the cummulant probability has it's most probable value and the contact point t^* would be the most probable contact point. So a good estimate for x_f would be found by setting the $t^* \cong 0$, from which we get

$$x_f \cong \sqrt{4tH} \quad (21)$$

We believe that the main contribution for the integral(15) would be around a characteristic value of H , say H^* such that the cumulative probability $F(H, [0, t])$ gives its dominant contribution. On the other hand we find H^* from

$$N(H, [0, t]) \cong 1 \quad (22)$$

or

$$\frac{\sqrt{\sigma_g^2 t'}}{S(t')\sqrt{2\pi}} e^{-\frac{S(t')^2}{2t'\sigma_g^2}} \cong 1 \quad (23)$$

Intuitively this characteristic value gives the border beyond which there would be just one intersection between the process $G(t')$ and the curve $S(t')$. After doing some algebra we find H^* in the large time limit and in the domain $x \ll x_f$ as

$$H^* \cong \sigma_g \sqrt{t \log t} \quad (24)$$

By substituting the derived H^* from (24) in expression (21) we find the time dependence of x_f

$$x_f^2 \cong \sigma_g t^{3/2} (\log t)^{1/2} \quad (25)$$

When $t \gg \sqrt{\frac{x^2 t}{4H}}$ or $x \ll x_f$ the cumulative probability density will be:

$$F(H, [0, t]) \cong e^{-\frac{\sigma_g \sqrt{t}}{\sqrt{2\pi}(H + \frac{x\sqrt{H}}{\sqrt{4t}})}} e^{-\frac{(H + \frac{x\sqrt{H}}{\sqrt{4t}})^2}{2\sigma_g^2 t}} \quad (26)$$

The asymptotic form of $F(H, [0, t])$ may be computed for large values of $H = H^* + \beta z$ and the result comes out to be the well known Gumbel distribution[12].

$$F(H, [0, t]) \cong e^{-e^{-z}} \quad (27)$$

where

$$\beta = \frac{\sigma_g^2 t}{H^* + \frac{x\sqrt{H^*}}{\sqrt{4t}}} \quad (28)$$

By substituting(16) in (15) and taking an integration by part we will get

$$P(t' \in [t', t' + \Delta t']) = \int_0^\infty N(H, [t_1, t_1 + \Delta t]) \frac{dF(H, [0, t])}{dH} dH \quad (29)$$

We have calculated the probability distribution in (29) by approximating $N(H, [t_1, t_1 + \Delta t])$ in (18) for deviations from H^* and substituting in (29):

$$P(t' \in [t', t' + \Delta t'] | x, t) = \gamma \Gamma(\alpha\beta + 1, 1) \quad (30)$$

Where α and γ are defined as

$$\gamma = \frac{\sqrt{2\sigma_g^2 t'}}{H^*} e^{\frac{1}{2\sigma_g^2 t'} (H^* + \frac{x^2}{4(t-t')})^2} \quad (31)$$

$$\alpha = \frac{S^*(t')}{\sigma_g^2 t'} \quad (32)$$

We have the following intuitive picture that as one measures the slope in a linear part of the velocity profile in a fixed spatial point the most probable value of is a decreasing function of x . By computing the numerical integration of the probability density one can easily see that for x 's greater x_f which is given by (21) the total probability of occuring a contact is vanishing. By changing variable from t' to $u(x, t)$, one can easily find the probability density of $u(x, t)$ for a fixed x and t as:

$$P(u|x, t) = \frac{x}{u^2 H^*} \sqrt{2\sigma_g^2 \left(t - \frac{x}{u}\right)} e^{-\frac{1}{2\sigma_g^2 \left(t - \frac{x}{u}\right)} \left(H^* + \frac{xu}{4}\right)^2} \Gamma\left(\frac{\left(H^* + \frac{xu}{4}\right)t}{H^* \left(t - \frac{x}{u}\right)} + 1, 1\right) \quad (33)$$

We have numerically integrated the average shock's slope and velocity for different values of x in a fixed observation time, and have plotted their character in different graphs (figures 1,2).

4 Two Point Probability Density

In this section we generalise the above idea for calculating the two point statistics of the shocks. By using the geometrical picture of the contacts between the hyperbola and the random process $G(t')$, every information about the two point statistics can be mapped to the different cathegories of geometrical events. If we are interested in the two point statistics in two different times t_1, t_2 but at the same fixed point x , the correspondent event would be the contact of two parabolas with different vertical asymptotic axes at t_1, t_2 , with the random process $G(t')$ in two points say t'_1, t'_2 without ever intersecting the other parts of it. Similarly the two point statistics in two different spatial points x_1, x_2 , and at a fixed observation time t , would be translated to definite geometrical events. Technically the same structure for calculating these quantities can be applied in principle. For example in calculating the two point probability density $P(u_1, u_2)$ on which $u_1 = u(x_1, t)$ and $u_2 = u(x_2, t)$, we use from the geometrical construction as in (fig 4). According to the figure, there are two situations distinguished which are in correspondence to existence and nonexistence of shocks between the two contact points t'_1 and t'_2 . So we proceed to find

$$P(t'_1 \in L_1, t'_2 \in L_2) = P_p(t'_1, t'_2) + \delta(t'_1 - t'_2) P_a(t'_1, t'_2) \quad (34)$$

where P_p corresponds to the peresence and P_a corresponds to absence of shocks between two points t'_1 and t'_2 . Existence of Dirac delta function in (34) is related to the fact that in large spatial scales $x \in [x_1, x_2]$ and in the absence of discontinuities between t'_1 and t'_2 the two hyperbolas should approximately contact near one of the local maxima of $G(t)$, thus $t'_1 \cong t'_2$.

By referring to (fig 3) and adapting the techniques of previous section for calculating $P_p(t'_1, t'_2)$ we write:

$$\begin{aligned} P_p(t'_1, t'_2) = & \int dH_1 \int dH_2 e^{-N(H_2, [0, t'_1])} e^{-N(H_2, [t'_1 + \Delta t, t^*])} e^{-N(H_1, [t^*, t'_2])} \times \\ & \times e^{-N(H_1, [t'_2 + \Delta t, t])} \frac{d}{dH_1} e^{-N(H_1, [t'_1, t'_1 + \Delta t])} \frac{d}{dH_2} e^{-N(H_2, [t'_2, t'_2 + \Delta t])} \end{aligned} \quad (35)$$

on which t^* is the intersection point of two hyperbolas, that is:

$$t - t^* = \frac{x_1^2 - x_2^2}{4(H_2 - H_1)} \quad (36)$$

It is clear that in the limit $\sigma_g t \gg H^*$ and for large times, the main contribution in (35) is related to region near H^* given in (24). Thus by defining $H_1 = H^* + \beta z_1$ and $H_2 = H^* + \beta z_2$ the integration of (35) will give the two point statistics. The main point in calculating the integral is the existence of the same level H^* over which the extremes of the proposed process $G(t)$ have their major contribution. Calculations in this direction will be pursued in future.

5 Diffusion Limit

One of the interesting questions is related to stationary state of the system and its relationship to our picture. We believe that the stationary regime in our problem is governed by the diffusion equation stirred at origin. For showing the validity of our statement we should compare the average of the two terms in the denominator of the expression for $u(x, t)$ given in (4). Before taking an integration by parts the denominator can be written as

$$c(0) \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) + \left(\frac{x}{\sqrt{\pi}}\right) \int_0^t dt' \frac{c(t')}{\sqrt{4\nu(t-t')^3}} e^{-\frac{x^2}{4\nu(t-t')}} \quad (37)$$

After taking the average over the distribution of $g(t)$ one can easily find that the average of $c(t)$ is

$$\langle c(t) \rangle = c(0) e^{\frac{\sigma_g^2 t}{2\nu^2}} \quad (38)$$

By substituting from (38) into the (37) and changing the variable of integration from t' to w we will end up with

$$c(0) \left[\operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) + \frac{x}{\sqrt{4\nu}} e^{\frac{\sigma_g^2 t}{2\nu^2}} \int_{\frac{1}{t}}^{+\infty} dw w^{-\frac{1}{2}} e^{-(\frac{x^2}{4\nu})w - (\frac{\sigma_g^2}{2\nu^2})(\frac{1}{w})} \right] \quad (39)$$

Because we are interested in steady state, we should find the large time limit of the denominator. The limit of the integral can be written in terms of the modified Bessel function $K_{\frac{1}{2}}\left[\frac{x\sigma_g}{\sqrt{2\nu^3}}\right]$, so the denominator in large time limit can be written as

$$c(0) e^{\frac{\sigma_g^2 t}{2\nu^2} (1 - \frac{x}{\sqrt{\nu t^*}})} \quad (40)$$

where

$$t^* = \frac{\sigma_g^2 t^2}{2\nu^2} \quad (41)$$

It is obvious that in the limit where $t \rightarrow +\infty$ the denominator behaves as (38) which means that the integral in the denominator of (4) would be negligible in average as if $u(x, t)$ satisfies the diffusion equation with random source at origin.

6 Conclusion

We have found the one point statistics of the Burgers equation with random boundary condition on a semiline. The random boundary condition in the problem can be translated as a random momentum input so the total momentum is not conserved. We have solved the problem by a geometrical construction which relates the original problem to the Extreme universality of the input noise in time([12, 13]). In the case of a non-homogeneous noise with a variance proportional with t we found the Gumbel universality class([12]). The overall picture which we have found is that in a snapshot of the velocity profile there is a moving point $x_{diff} \cong \sqrt{\nu t^*}$ which gives a crossover region from the diffusive behaviour to shock structure such that with the passage of the time the diffusive region will grow and in the infinite time limit the diffusive regime would be dominant in the whole semiline.

Recently it has been shown that there are some evidences which relates different Extreme universalities with different Replica symmetry breaking schemes([13, 14]) and specifically the relation between the Gumbel universality with to the one step Replica symmetry breaking is checked in some models. Checking another paradigm we think that it would be important to apply the Replica method on this problem too. Another problem worthy of investigation is the calculation of intermittency exponent structure function, and checking whether the nonhomogeneity alters the intermittency exponent. Finally we think that the KPZ and directed polymer analogs of our problem which have been studied recently([15]) is relevant too because the known results in those realms are studied under the homogeneous noise and the presence of a nonhomogeneous noise might change the universality classes in those problems.

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Figure Captions

Figure 1- The behaviour of average shock slope in terms of x . $\sigma_g = 1$ and the observation time is $t = 1000$. The best fit for the curve is $\langle \lambda \rangle \cong \frac{1}{a+bx^{3/2}}$ where $a \cong 182$ and $b \cong 0.05$.

Figure 2- The average velocity in terms of x . $\sigma_g = 1$ and the observation time is $t = 1000$. The best fit for the curve is $\ln \langle u(x) \rangle \cong a + b\frac{\ln x}{x}$ where $a \cong -0.1$ and $b \cong -16$.

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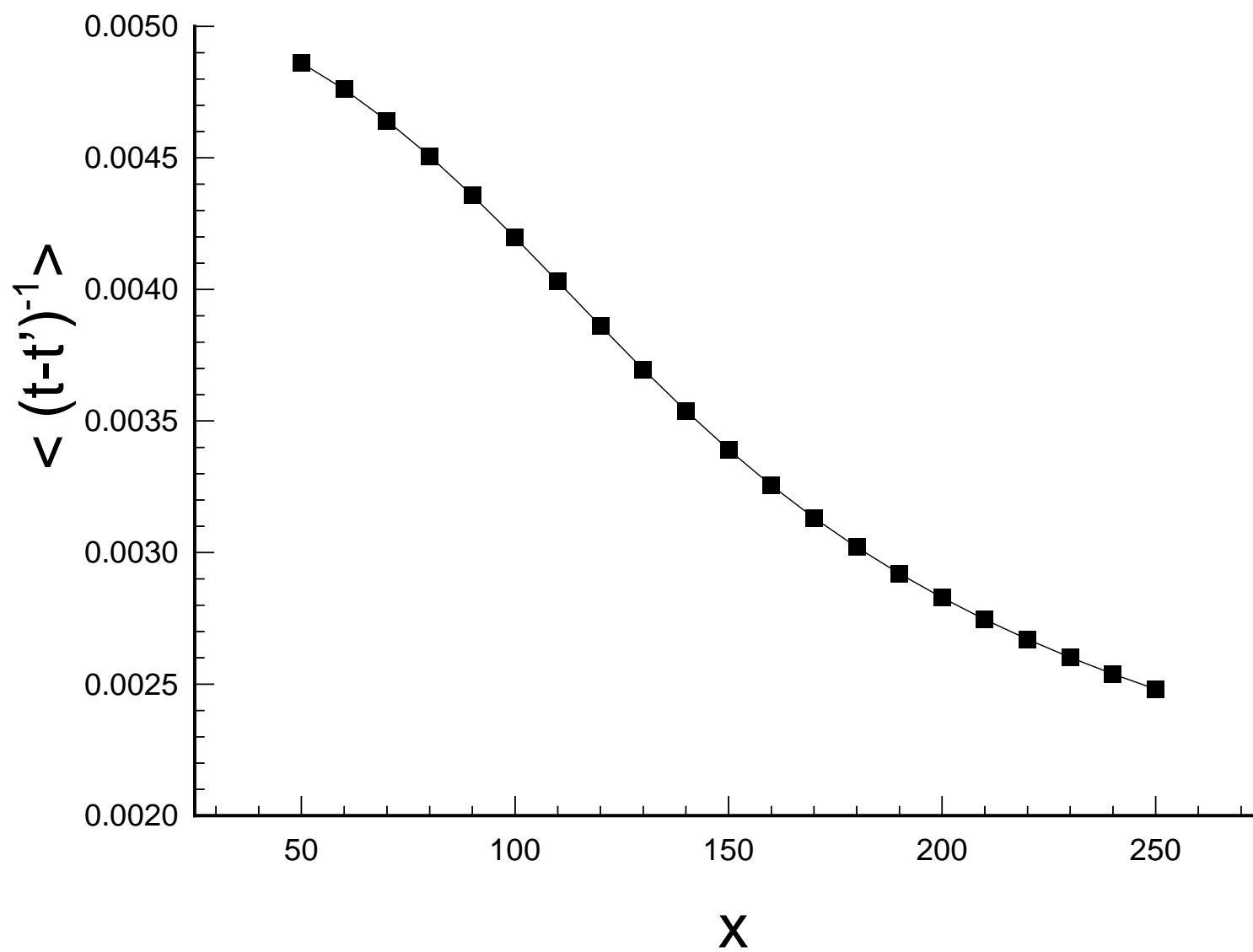


Figure.1

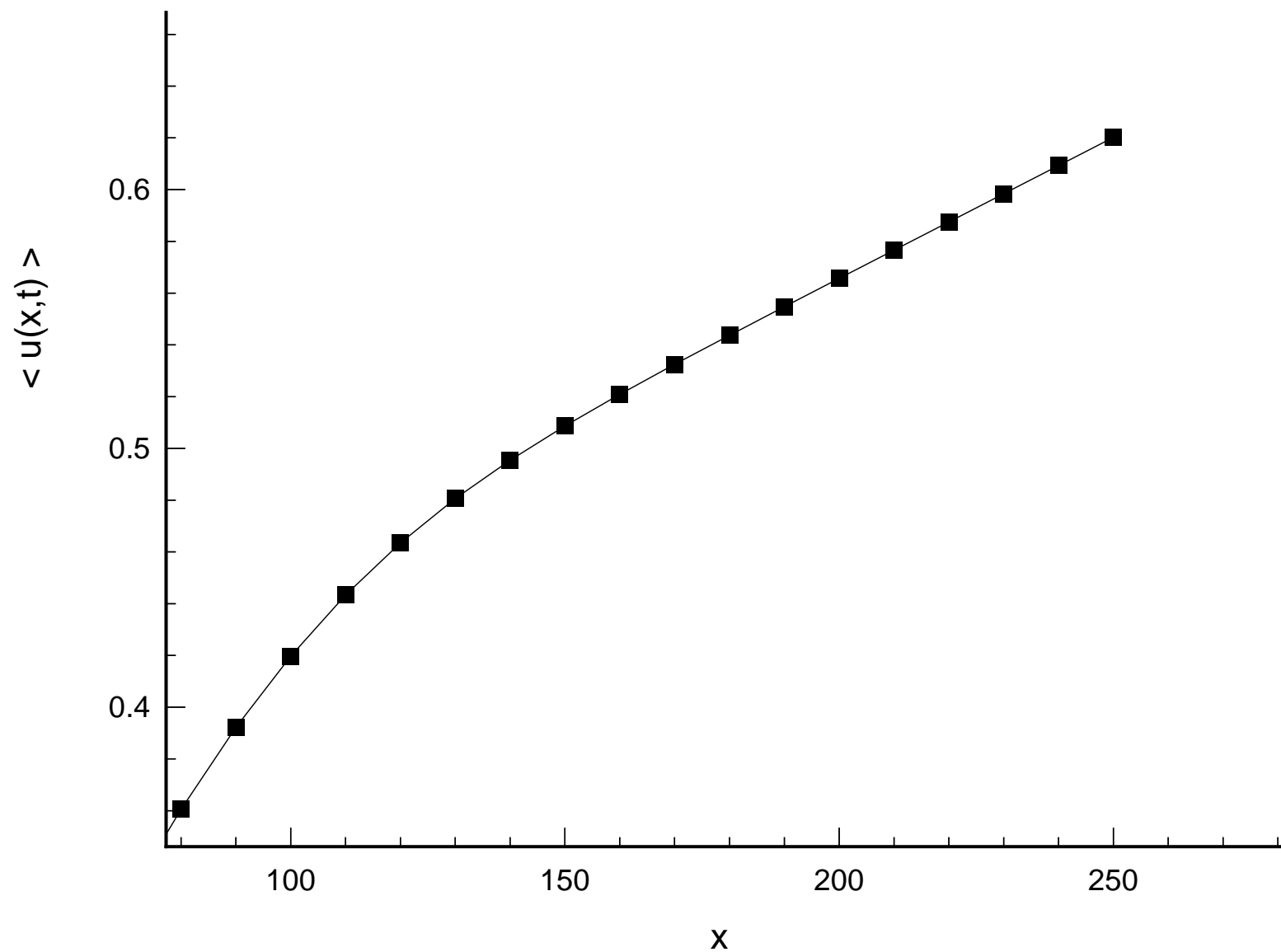


Figure.2